# On the Behavior of the Mixed-Integer SMS-EMOA on Box-Constrained Quadratic Bi-Objective Models

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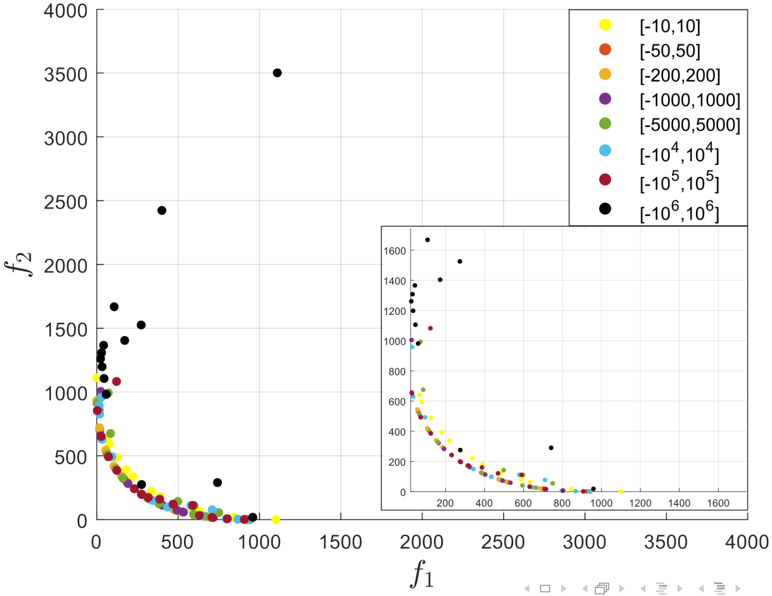
# Motivation: White-Box Solvers' Deficiency



# MIQP

A QP is *standardly* formed to minimize a quadratic objective function whose decision variables are within the unit simplex in $\mathbb{R}_+^D$:

$$\begin{aligned} &\text{minimize}_{\vec{x}\in\Delta_D} \vec{x}^T \mathbf{H} \vec{x} \\ &\Delta_D := \left\{ \vec{x} \in \mathbb{R}_+^D : \vec{e}^T \vec{x} = 1 \right\}, \end{aligned} \tag{1}$$

$\mathbf{H}$ is a symmetric $D \times D$ real matrix, and $\vec{e} \in \mathbb{R}^D$ is the vector of ones.

**MIQP**: the $D$-dimensional decision vector $\vec{x}$ is constructed by $n_r$ real-valued decision variables followed by $n_z$ integer decision variables that are defined by so-called the *index set*:

$$I := \{n_r + 1, \ldots, n_r + n_z\} : \quad \forall i \in I \quad x_i \in \mathbb{Z}. \tag{2}$$

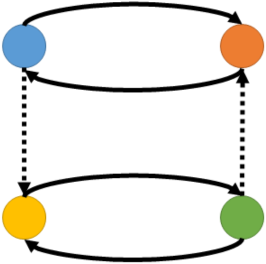
# complexity of unbounded problems

**theorem** [Jeroslow, 1973]
a quadratically-constrained mixed-integer unbounded problem is undecidable.

1. a generalized unbounded MIQP **cannot be linearized** (even if the integers $x_i \ : \ i \in I$ may be linearized using auxiliary binaries, the multiplication of two (loosely bounded) decision variables within $\vec{x}$ cannot be precisely linearized (rather could be piecewise approximated upon separation).
2. despite this theoretical complexity, there has been **much practical progress in treating Quadratically-Constrained problems** (e.g., by reformulation to a bilinear programming problem with integer variables, or by diverting to Mixed-Integer Second-Order Cone Programming when the model permits).

# research question

*What is the algorithmic behavior of the SMS-EMOA with mixed-integer evolution strategies based mutation operators when treating bi-objective under-constrained MIQP models? Especially, how capable are they to handle loose boundary conditions, and what are the evolutionary operators that enable their convergence?*



methodology

# problem formulation

We consider the following family of MI objective functions:

$$f_k(\vec{x}) := (\vec{x} - \vec{\xi}_k)^T \cdot \mathcal{H}_k \cdot (\vec{x} - \vec{\xi}_k), \tag{3}$$

where the $D$-dimensional decision vector $\vec{x}$ is constructed by $n_r$ real-valued decision variables followed by $n_z$ integer decision variables that are defined by so-called the *index set*

$$I := \{n_r + 1, \dots, n_r + n_z\} : \quad \forall i \in I \quad x_i \in \mathbb{Z}.$$

We target the Pareto optimization of this family of unconstrained bi-objective convex quadratic problems:

$$\boxed{\begin{array}{l} f_0(\vec{x}) \to \min, \quad f_1(\vec{x}) \to \min \\ \text{subject to:} \\ \vec{x} \in \mathbb{R}^D, x_i \in D_i \subseteq \mathbb{Z} \quad \forall i \in I \end{array}} \tag{4}$$

with $I$ being the integers' index set.

# MIQP instances

We consider 3 Hessian matrices for unconstrained and box-constrained problems with dimensions $n = n_r = n_z = D/2$ (**with** $c$ **denoting a parametric condition number**):

**H-1** Ellipse: $(\mathcal{H}_{\text{ellipse}})_{ii} = c^{\frac{i-1}{n-1}}$;

**H-2** Rotated Ellipse (RotEllipse): $\mathcal{H}_{\text{RE}} = \mathcal{R}\mathcal{H}_{\text{ellipse}}\mathcal{R}^{-1}$, where $\mathcal{R}$ is rotation by $\approx \frac{\pi}{4}$ radians in the plane spanned by $(1, 0, 1, 0, \ldots, 1, 0)^T$ and $(0, 1, 0, 1, \ldots, 0, 1)^T$;

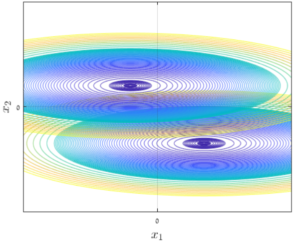
**H-3** Hadamard Ellipse (HadEllipse): $\mathcal{H}_{\text{HE}} = \mathcal{S}\mathcal{H}_{\text{ellipse}}\mathcal{S}^{-1}$, where the rotation constitutes the normalized Hadamard matrix, $\mathcal{S} := \text{Hadamard}(D)/\sqrt{D}$.

We set two points about which the quadratic models are centered:

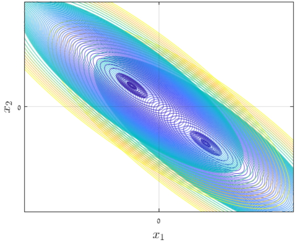
$$\vec{\xi}_0 := (+7, -7, +7, -7, \ldots, +7, -7)^T$$
$$\vec{\xi}_1 := (-4, +4, -4, +4, \ldots, -4, +4)^T \quad .$$

# testbed

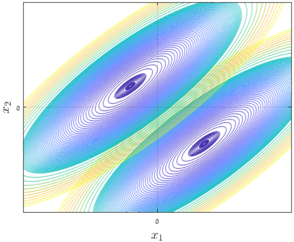
ELLIPSE $c = 10$



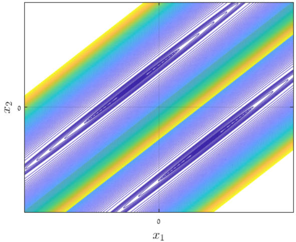

ROTELLIPSE $c = 10$



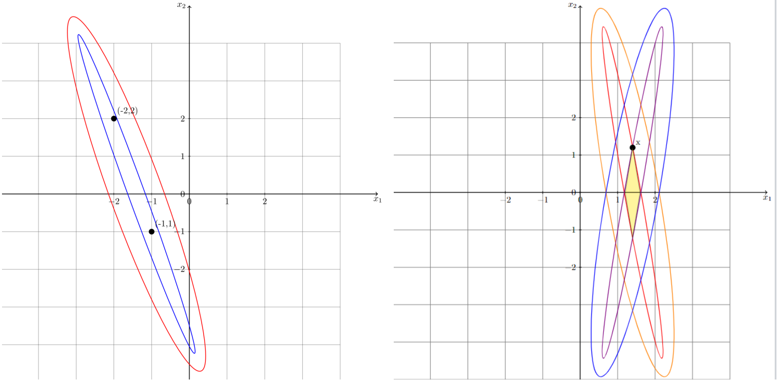

HADELLIPSE $c = 10$




HADELLIPSE $c = 1000$



# integer challenges




LHS: Possible contour lines for the problems ROTELLIPSE or HADELLIPSE. The search-point $(-2, 2)$ can only be improved by a move to $(-1, -1)$ which is not a neighboring point on the integer lattice. RHS: Contours of 2 functions over a continuous domain; the yellow area: points dominating $x$.

# SMS-EMOA

**SMS-EMOA**

```
1: t ⟵ 0
2: P_t ⟵ initPopulation()
3: repeat
4:     Q_t ⟵ P_t ∪ generateOffspring(P_t)
5:     {R_1, ..., R_w} ⟵ nonDominatedSort(Q_t)
6:     k ⟵ argmin_{s∈R_w} [ΔV(s, R_w)]
7:     P_{t+1} ⟵ (Q_t \ {k})
8:     t ← t + 1
9: until termination condition satisfied
10: return {P_t}
```

```
mutate(x⃗, s⃗, n_r, z⃗, q⃗, n_z)
```

$\texttt{mutate}(\vec{x},\ \vec{s},\ n_r,\ \vec{z},\ \vec{q},\ n_z)$

**/* real-valued decision variables */**

$\mathcal{N}_g^{(r)} \leftarrow \mathcal{N}(0,1)\,,\ \tau_g^{(r)} \leftarrow \frac{1}{\sqrt{2 \cdot n_r}},\ \tau_\ell^{(r)} \leftarrow \frac{1}{\sqrt{2 \cdot \sqrt{n_r}}}$

**for** $i = 1, \ldots, n_r$ **do**

$s_i' \longleftarrow \max\left(\varepsilon,\ s_i \cdot \exp\left\{\tau_g^{(r)} \cdot \mathcal{N}_g^{(r)} + \tau_\ell^{(r)} \cdot \mathcal{N}(0,1)\right\}\right)$

$x_i' \longleftarrow x_i + \mathcal{N}(0, s_i')$

**end**

**/* integer decision variables */**

$\mathcal{N}_g^{(z)} \leftarrow \mathcal{N}(0,1)\,,\ \tau_g^{(z)} \leftarrow \frac{1}{\sqrt{2 \cdot n_z}},\ \tau_\ell^{(z)} \leftarrow \frac{1}{\sqrt{2 \cdot \sqrt{n_z}}}$

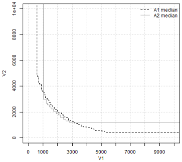
**for** $i = 1, \ldots, n_z$ **do**

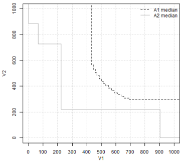
$q_i' \longleftarrow \max\left(1,\ q_i \cdot \exp\left\{\tau_g^{(z)} \cdot \mathcal{N}_g^{(z)} + \tau_\ell^{(z)} \cdot \mathcal{N}(0,1)\right\}\right)$

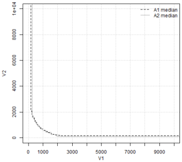
$z_i' \longleftarrow z_i + \mathcal{G}_{n_z}(0, q_i')$ // Geometrically-distributed !

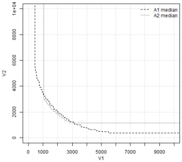
**end**

**return** $\{\vec{x}',\ \vec{s}',\ \vec{z}', \vec{q}'\}$



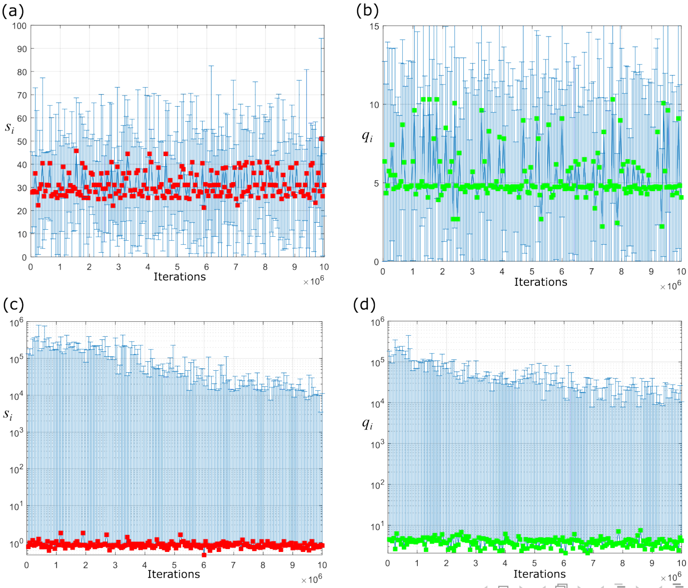




# observations

# 0. baseline runs with tight box-constraints

- Ellipse: SMS-EMOA attained fine approximations to the Pareto frontier (equally good as the DMA).
- RotEllipse: SMS-EMOA attained fine results for the $c = 10$ instance, but is no longer capable of finding a good approximation as the conditioning increases (unlike the DMA which performs well up to $c = 10^6$).
- HadEllipse: SMS-EMOA exhibits poor performance regardless of the condition number. May be explained by the narrow cone of dominating solutions.

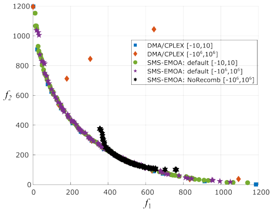
# 1. mutation and step-sizes' adaptation



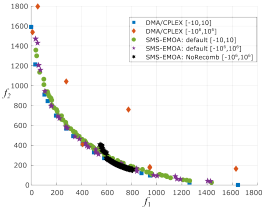
# 2. the recombination operator

We suspended this MIES recombination and observed the difference w.r.t. the default-SMS or the white-box DMA.

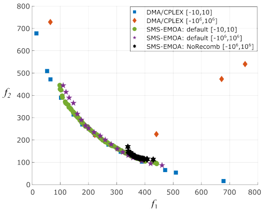
ELLIPSE $c = 1000$




ROTELLIPSE $c = 100$

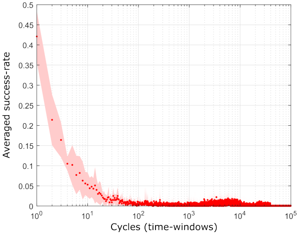



HADELLIPSE $c = 10$

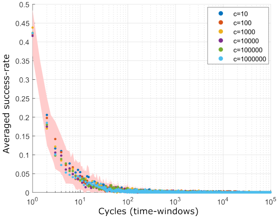


# 3. the success-rate

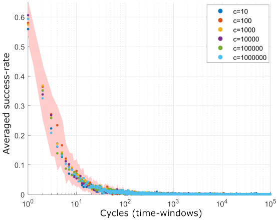
## Sphere reference:



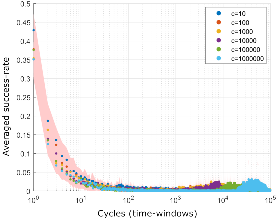

## Ellipse

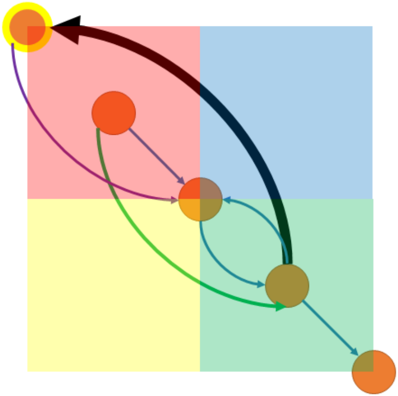



## RotEllipse




## HadEllipse





discussion

# summary and take-home messages

- BBO remedies WBO's deficiency on loosley bounded cases!
- There is no step-size convergence in the tight scenario (median values stagnate). In the loose scenario, the dramatic large standard deviations of the step-sizes gradually shrink upon convergence — indicative for the MIES' capability to handle the enormous bounding box of $[-10^6, 10^6]^{64}$.
- Upon suspending the recombination operator, we concluded that *the mutation operator is responsible for the MIES' ability to handle the loosely bounded decision variables (via self-adaptation), and to enable a focused search in a tight regime. At the same time, without the recombination operator, SMS-EMOA was less potent to explore tradeoff areas and results in a deteriorated coverage.*
- Investigation of the empirical success-rate revealed that the major progress toward the Pareto frontier was made during the first stage of the optimization process. The apparent lack of progress in the later stages of is a weakness ($\Rightarrow$ an opportunity!).

Ofer Shir acknowledges the Faculty of Mathematics at the Technion - Israel Institute of Technology for hosting him during his sabbatical.

**obrigado**