Introductory Mathematical Programming for EC

Ofer M. Shir (Tel-Hai College & Migal Institute, ISRAEL)

ofersh@telhai.ac.il

Tel-Hai College

MIGAL Galilee Research Institute LTD.

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about the presenter

Ofer Shir is an Associate Professor of Computer Science at Tel-Hai College and a Principal Investigator at the Migal Research Institute – Upper Galilee, ISRAEL.

Previously:

- IBM-Research
- Princeton University: Postdoctoral Research Associate
- PhD in CS: Leiden-U adv. : Th. Bäck & M. Vrakking; BSc in CS&Phys at Hebrew-U

why are we here?

- Global optimization has been for several decades addressed by algorithms and Mathematical Programming (MP) — branded as Operations Research (OR), yet rooted at Theoretical CS [1].
- Also – it has been treated by dedicated heuristics ("Soft Computing") – where EC resides (!)
- These two branches complement each other, yet practically studied under two independent CS disciplines

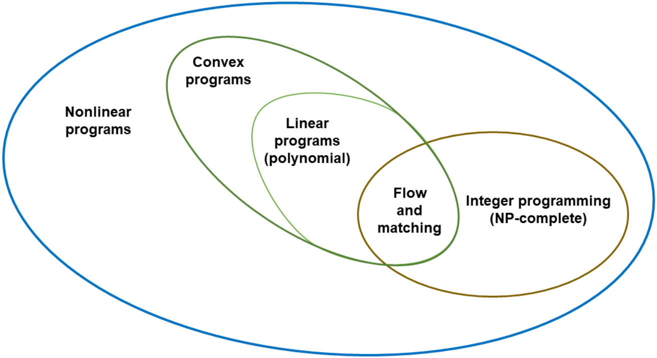
further motivation

EC scholars become stronger, better-equipped researchers when obtaining knowledge on this so-called "optimization complement"

Commonly-encountered **misbeliefs**:

- *"if the problem is non-linear, there is no choice but to employ a Randomized Search Heuristic"*
- *"if it's a combinatorial NP-complete problem, EAs are the most reasonable option to approach it"*
- *"neither Pareto optimization nor uncertainty is/are addressed by OR"*
- *"OR is the art of giving bad answers to problems, to which, otherwise worse answers are given"*

outline



Mathematical Programming: fundamentals

based on (i) MIT's "Optimization Methods" course material by D. Bertsimas, (ii) "Combinatorial Optimization" by Ch. Papadimitriou & K. Steiglitz, (iii) "The Nature of Computation" by C. Moore and S. Mertens, and (iv) IBM's ILOG/OPL tutorials and documentation.

the field of operations research

- Developed during WW-II: mathematicians assisted the US-army to solve hard strategical and logistical problems; mainly planning of operations and deployment of military resources. Due to the strong link to military *operations*, the term *Operations Research* was coined.
- Post-war: knowledge transfer into industry
- Roots: linear programming (LP), pioneered by George B. Dantzig
- Dantzig worked for the US-government, formulating the generalized LP problem, and devising the Simplex algorithm for tackling it. He also pursued an academic career (Berkeley, Stanford).

mathematical optimization

- Partitioning into 2 main approaches: constraints programming (CP) *versus* mathematical programming (MP). CP is concerned with constraints satisfaction problems, which possess no explicit objective functions (sometimes because impossible to model)
- MP includes the following techniques:
  1. linear programming (LP)
  2. integer programming (IP)
  3. mixed-integer programming (MIP)
  4. quadratic programming (QP) and mixed-integer QP (MIQP)
  5. nonlinear programming (NLP)

roots of continuous optimization

- **Fermat, 1638; Newton 1670**

$$\min_{x} \; f(x) \qquad x \in \mathbb{R}$$

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = 0$$

- **Euler, 1755**

$$\min_{\vec{x}} \; f(\vec{x}) \qquad \vec{x} \in \mathbb{R}^d$$

$$\nabla f(\vec{x}) = 0$$

- **Lagrange, 1797**

$$\min_{\vec{x}} \; f(\vec{x}) \qquad \vec{x} \in \mathbb{R}^d$$

$$\text{subject to: } g_k(\vec{x}) = 0 \qquad k = 1, \ldots, m$$

- **Euler, Lagrange**: infinite dimensions, calculus of variations

the canonical optimization problem

The general nonlinear problem formulated in the canonical form [2]:

$$\boxed{\begin{array}{ll} \text{minimize}_{\vec{x}}\ f(\vec{x}) & \qquad \vec{x} \in \mathbb{R}^d \\ \text{subject to: } g_1(\vec{x}) \geq 0 & \\ \qquad\qquad\quad \vdots & \\ \qquad\qquad\quad g_m(\vec{x}) \geq 0 & \\ \qquad\qquad\quad h_1(\vec{x}) = 0 & \\ \qquad\qquad\quad \vdots & \\ \qquad\qquad\quad h_\ell(\vec{x}) = 0 & \end{array}} \tag{1}$$

solving the general problem

- Convexity:
  1. $f : \mathcal{S} \to \mathbb{R}$
  2. The function is convex **iff** $\forall s_1, s_2 \in \mathcal{S},\ \ 0 < \lambda < 1$

$$f\left(\lambda s_1 + (1-\lambda)\, s_2\right) \leq \lambda f\left(s_1\right) + (1-\lambda)\, f\left(s_2\right)$$

  3. f is concave if $-f$ is convex.
- The problem is called a *convex programming problem* when
  - i f is convex
  - ii g_i are all concave
  - iii h_j are all linear

- Strongest property: local optimality implies global optimality
- Sufficient conditions for optimality exist (Kuhn-Tucker)

linear programming: standard form

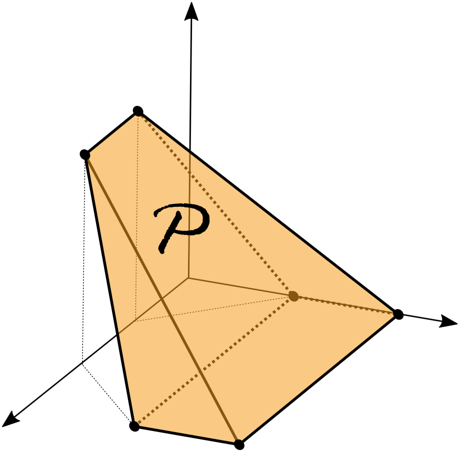
When f and the constraints are all linear, LP is formed by the **standard form** (minimization, equality constraints, non-negative variables) to search over a d-dimensional space, $\vec{x} \in \mathbb{R}^d$:

$$\boxed{\begin{aligned} &\text{minimize}_{\vec{x}}\ \vec{c}^T \vec{x} \\ &\text{subject to: } \mathbf{A}\vec{x} = \vec{b} \\ &\qquad\qquad\quad \vec{x} \geq 0 \end{aligned}} \tag{2}$$

with $\mathbf{A} \in \mathbb{R}^{m \times d}$ and $\vec{b} \in \mathbb{R}^m$ describing the constraints.

polyhedra

- A **hyperplane** is defined by the set
$$\left\{\vec{x} \in \mathbb{R}^d : \vec{a}^T\vec{x} = b_0\right\}$$
- A **halfspace** is defined by the set
$$\left\{\vec{x} \in \mathbb{R}^d : \vec{a}^T\vec{x} \geq b_0\right\}$$
- A **polyhedron** is constructed by the intersection of many halfspaces.
- The finite set of candidate solutions is the set of vertices of the **convex polyhedron** (*polytope*) defined by the linear constraints!
- Thus, solving any LP reduces to selecting a solution from a finite set of candidates ⇒ the problem is **combinatorial** in nature.



geometry of LP

Given a *polytope*

$$\mathcal{P} := \left\{\vec{x} \in \mathbb{R}^d : \mathbf{A}\vec{x} \leq \vec{b}\right\}$$

- The point $\vec{x}$ is a **vertex** of $\mathcal{P}$
- $\vec{x} \in \mathcal{P}$ is an **extreme point** of $\mathcal{P}$ if

$$\nexists \vec{y}, \vec{z} \in \mathcal{P} \, (\vec{y} \neq \vec{x}, \vec{z} \neq \vec{x}) : \quad \vec{x} = \lambda \vec{y} + (1 - \lambda) \, \vec{z}, \; 0 < \lambda < 1$$

- $\vec{x} \geq \vec{0} \in \mathbb{R}^d$ is a **basic feasible solution** (**BFS**) iff $\mathbf{A}\vec{x} = \vec{b}$ and exist indices $\mathcal{B}_1, \ldots, \mathcal{B}_m$ such that:
  (i) the columns $\mathbf{A}_{\mathcal{B}_1}, \ldots, \mathbf{A}_{\mathcal{B}_m}$ are linearly independent
  (ii) if $\jmath \neq \mathcal{B}_1, \ldots, \mathcal{B}_m$ then $x_\jmath = 0$

polytopes and LP

"Corners" definitions: equivalence theorem

$\mathcal{P} := \left\{\vec{x} \in \mathbb{R}^d : \mathbf{A}\vec{x} \leq \vec{b}\right\}$; let $\vec{x} \in \mathcal{P}$.

$$\vec{x} \text{ is a vertex } \iff \vec{x} \text{ is an extreme point } \iff \vec{x} \text{ is a BFS}$$

See, e.g., [3] for the proof.

Conceptual LP search:

- begin at any "corner"
- **while "corner" is not optimal** hop to its neighbouring "corner" as long as it improves the objective function value

the basic simplex

$t \leftarrow 0$; *opt*, *unbounded* $\leftarrow$ `false`, `false`

$\vec{x}_t \leftarrow$ `constructBFS()`, $\mathbf{B} \leftarrow [\mathbf{A}_{\mathcal{B}_1}, \ldots, \mathbf{A}_{\mathcal{B}_m}]$

while *!opt && !unbounded* **do**

    if $\bar{c}_j := c_j - \vec{c}_B^T \mathbf{B}^{-1} \mathbf{A}_j \geq 0 \quad \forall j$ **then** *opt* $\leftarrow$ `true`

    else

        select any j such that $\bar{c}_j < 0$

        if $\vec{u} := \mathbf{B}^{-1} \mathbf{A}_j \leq \vec{0}$ **then** *unbounded* $\leftarrow$ `true`

        else

            $\vec{x}_{t+1} \leftarrow$ pivot on $\vec{x}_t$ `/* see [4] for details */`

            set new basis $\mathbf{A}_j$ `/* see [4] for details */`

            $t \leftarrow t + 1$

        end

    end

end

output: $\vec{x}_t$

duality

i. Every LP has an associated problem known as its **dual**; min turns into max, each constraint in the primal has an associated dual variable:

$$\begin{aligned} \text{minimize}_{\vec{x}} \quad & \vec{c}^T\vec{x} \quad \vec{x}\in\mathbb{R}^d \\ \text{subject to: } & \mathbf{A}\vec{x}=\vec{b} \\ & \vec{x}\geq 0 \end{aligned} \qquad \begin{aligned} \text{maximize}_{\vec{p}} \quad & \vec{p}^T\vec{b} \quad \vec{p}\in\mathbb{R}^m \\ \text{subject to: } & \vec{p}^T\mathbf{A}\leq\vec{c}^T \end{aligned}$$

$$\begin{aligned} \text{minimize}_{\vec{x}} \quad & \vec{c}^T\vec{x} \quad \vec{x}\in\mathbb{R}^d \\ \text{subject to: } & \mathbf{A}\vec{x}\geq\vec{b} \end{aligned} \qquad \begin{aligned} \text{maximize}_{\vec{p}} \quad & \vec{p}^T\vec{b} \quad \vec{p}\in\mathbb{R}^m \\ \text{subject to: } & \vec{p}^T\mathbf{A}=\vec{c}^T \\ & \vec{p}\geq 0 \end{aligned}$$

ii. The dual of the dual is the primal.

duality theorems [von Neumann, Tucker]

- **Weak duality theorem**
  If $\vec{x} \in \mathbb{R}^d$ is primal feasible and $\vec{p} \in \mathbb{R}^m$ is dual feasible then

$$\vec{p}^T \vec{b} \leq \vec{c}^T \vec{x}$$

- Corollary: If $\vec{x}$ is primal feasible, $\vec{p}$ is dual feasible, and $\vec{p}^T \vec{b} = \vec{c}^T \vec{x}$, then $\vec{x}$ is optimal in the primal and $\vec{p}$ is optimal in the dual.

- **Strong duality theorem**
  Given an LP, if it has an optimal solution – then so does its dual – having equal objective functions' values.

$\Rightarrow$ **The dual provides a bound that in the best case equals the optimal solution to the primal – and thus can help solve difficult primal problems.**

dual simplex

- Simplex is a primal algorithm: maintaining primal feasibility while working on dual feasibility
- Dual-simplex: maintaining dual feasibility while working on primal feasibility –
  Implicitly use the dual to obtain an optimal solution to the primal as early as possible, regardless of feasibility; then hop from one vertex to another, while gradually decreasing the infeasibility while maintaining optimality
- **Dual-simplex is the first practical choice for most LPs**.

R. Vanderbei, *Linear Programming: Foundations and Extensions*. Springer, 5^{th} ed., 2020, ISBN: 978-3-030-39414-1.

simplex: convergence

- Dantzig's simplex finds an optimal solution to any LP in a finite number of steps (avoiding cycles is easy, but not mentioned).
- Over half-century of improvements, its robust forms are very effective in treating very large LPs.
- However, simplex is not a polynomial-time algorithm, even if it is fast in practice over the majority of cases.
- *Pathological* LP-cases exist (e.g., the Klee-Minty cube [5]) – where an **exponential number of steps** is needed for convergence.
- An **ellipsoid algorithm** [5], devised by Soviet mathematicians in the late 1970's, is guaranteed to solve every LP in a polynomial number of steps.

“high-level” ellipsoid [Shor-Nemirovsky-Yudin]

input : a bounded convex set $\mathcal{P} \in \mathbb{R}^d$

$t \leftarrow 0$

$\mathcal{E}_t \leftarrow$ ellipsoid containing $\mathcal{P}$

while *center* $\vec{\xi}_t$ *of* $\mathcal{E}_t$ *is not in* $\mathcal{P}$ **do**

  let $\vec{c}^T\vec{x} \leq \vec{c}^T\vec{\xi}_t$ be such that $\left\{\vec{x}:\ \vec{c}^T\vec{x} \leq \vec{c}^T\vec{\xi}_t\right\} \supseteq \mathcal{P}$

  update to the ellipsoid with minimal volume containing the intersected subspace:

$$\mathcal{E}_{t+1} \leftarrow \mathcal{E}_t \cap \left\{\vec{x}:\ \vec{c}^T\vec{x} \leq \vec{c}^T\vec{\xi}_t\right\}$$

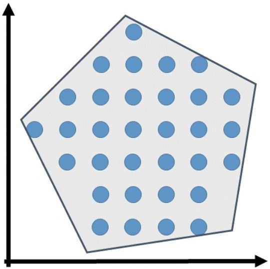
  $t \leftarrow t+1$

end

output: *center* $\vec{\xi}_t \in \mathcal{P}$

ellipsoid aftermath

- Polynomial-time algorithm for obtaining $\vec{x}^*$ within any given bounded convex set
- Khachian first used it (1979) to show polynomial solvability of LPs
- **Theorem**: if there exists a polynomial-time algorithm for solving a strict linear inequalities problem, then there exists a polynomial-time algorithm for solving LPs (see [3] for the proof).
- Conceptual novelty: disregarding the combinatorial nature of LPs
- In practice, unlike simplex, the ellipsoid is slow yet steady.
- However, its theoretical "polynomiality" has strong implications also for discrete optimization.



discrete optimization

roots of combinatorial optimization

Schrijver explored the history of combinatorial optimization:

- **Assignment: Monge, 1784** the assignment problem is one of the first discrete optimization problems to be investigated:

$$\textbf{[assignment]} \quad \text{minimize} \sum_{i=1}^{d} c_{i,\pi(i)} \tag{3}$$

  where $(c_{ij}) \in \mathbb{R}^{d \times d}$ is the cost matrix, and the search is over permutations π of order d.
- **Bipartite matching: Frobenius, ∼1912; König, ∼1915**
- **Transportation/supply-chain: Tolstoĭ, 1930**

A. Schrijver, "On the history of combinatorial optimization (till 1960)".

from LP to ILP

- The introduction of integer decision variables into a linear optimization problem yields a so-called (mixed)-integer linear program ((M)ILP) [6].
- A powerful modeling framework with much flexibility in describing discrete optimization problems
- The general ILP is itself *NP-complete* — and yet, there are subsets of “very easy” versus “very hard” problems
- *p2p shortest path* over a graph with d nodes has an $\mathcal{O}(d^2)$ algorithm, versus the *traveling salesman problem*...
- Unlike “pure-LP”, whose complexity is dictated by $d + m$ (variables+constraints), the choice of formulation in ILP is critical!
- Direction — what if the **constraint matrix** is **unimodular** [7] ?

integer linear optimization

- Pure integer:

$$\boxed{\begin{array}{ll} \text{maximize}_{\vec{x}} & \vec{c}^T\vec{x} \\ \text{subject to:} & \mathbf{A}\vec{x} \leq \vec{b} \\ & \vec{x} \in \mathbb{Z}_+^d \end{array}} \tag{4}$$

- Binary optimization (**important special case**):

$$(4) \text{ with } \vec{x} \in \{0,1\}^d$$

- Mixed-integer:

$$\boxed{\begin{array}{ll} \text{maximize}_{\vec{x}} & \vec{c}^T\vec{x} + \vec{h}^T\vec{y} \\ \text{subject to:} & \mathbf{A}\vec{x} + \mathbf{B}\vec{y} \leq \vec{b} \\ & \vec{x} \in \mathbb{Z}_+^d,\ \vec{y} \in \mathbb{R}_+^\ell \end{array}} \tag{5}$$

LP relaxations and the convex hull

- Given a discrete optimization problem, its consideration as a "*pure*" (continuous) LP is called its **LP relaxation**; e.g., each binary variable becomes continuous within the interval $[0, 1]$:

$$x_i \in \{0, 1\} \;\rightsquigarrow\; 0 \leq x_i \leq 1$$

- Formally, given a valid ILP formulation $\left\{\vec{x} \in \mathbb{Z}^d_+ \mid \mathbf{A}\vec{x} \leq \vec{b}\right\}$, the polytope $\left\{\vec{x} \in \mathbb{R}^d \mid \mathbf{A}\vec{x} \leq \vec{b}\right\}$ constitutes its LP relaxation.
- The **convex hull** of a set of points is defined as the "smallest polytope" that contains all of the points in the set; given a finite set $S := \left\{p^{(1)}, \ldots, p^{(N)}\right\}$, it is defined as

$$\mathcal{C}(S) := \left\{ q \;\middle|\; q = \sum_k^N \lambda_k p^{(k)} \;,\; \sum_k^N \lambda_k = 1,\; \lambda_k \geq 0,\; p^{(k)} \in S \right\} \quad (6)$$

- The **integral hull** is the *convex hull of the set of integer solutions*:

$$\widetilde{\mathcal{P}} := \mathcal{C}(X), \quad X \subset \mathbb{Z}^d \text{ solution points}$$

quality of formulations

- The quality of an ILP formulation for a problem having a feasible solution set X, is governed by the **closeness** of the *feasible set of its LP relaxation* to $\mathcal{C}(X)$.
- Given an ILP with two formulations, $\{P_1,\ P_2\}$, let $\left\{P_1^{LR},\ P_2^{LR}\right\}$ denote the feasible sets of their LP relaxations:
  we state that P_1 **is as strong as** P_2 if $P_1^{LR} \subseteq P_2^{LR}$, or that P_1 **is better than** P_2 if $P_1^{LR} \subset P_2^{LR}$ (strictly).
- If the *integral hull* is attainable as $\widetilde{\mathcal{P}} = \left\{\vec{x} \in \mathbb{R}^d \mid \widetilde{\mathbf{A}}\vec{x} \leq \widetilde{\vec{b}}\right\}$, the problem is polynomially solvable (all vertices are integers!) [6]
- Another perspective: an LP relaxation of an ILP with a totally unimodular constraint matrix will have only integer solutions [7].
- "**Easy Polyhedra**": MILP with fully-understood integral hulls — *assignment, min-cost flow, matching, spanning tree, etc.*

branch-and-bound

One of the common approaches to address integer programming, relying on the ability to bound a given problem.

It is a tree-search, adhering to the principle of *divide-and-conquer*:

(i) **branch**: select an active subproblem $\hat{\mathcal{F}}$
(ii) **prune**: if $\hat{\mathcal{F}}$ is infeasible – discard it
(iii) **bound**: otherwise, compute its lower bound $L(\hat{\mathcal{F}})$
(iv) **prune**: if $L(\hat{\mathcal{F}}) \geq U$, the current best upper bound, discard $\hat{\mathcal{F}}$
(v) **partition**: if $L(\hat{\mathcal{F}}) < U$, either completely solve $\hat{\mathcal{F}}$, or further break it to subproblems added to the list of active problems

"high-level" LP-based branch-and-bound

input : a linear integer program $\mathcal{F}$

$\Omega \leftarrow \{\mathcal{F}\}$; $U \leftarrow \infty$ /* active problems' set; global upper bound */

while Ω *is not empty* **do**

  let $\hat{\mathcal{F}}$ be a active subproblem, $\hat{\mathcal{F}} \in \Omega$; $\Omega \leftarrow \Omega \backslash \left\{\hat{\mathcal{F}}\right\}$

  compute its lower bound $L(\hat{\mathcal{F}})$ **by solving its LP relaxation**

  if $L(\hat{\mathcal{F}}) < U$ **then**

    $U \leftarrow L(\hat{\mathcal{F}})$

    if *exists heuristic solution* $\vec{\psi}$ *for* $\hat{\mathcal{F}}$ **then** $\vec{x}^* \leftarrow \vec{\psi}$

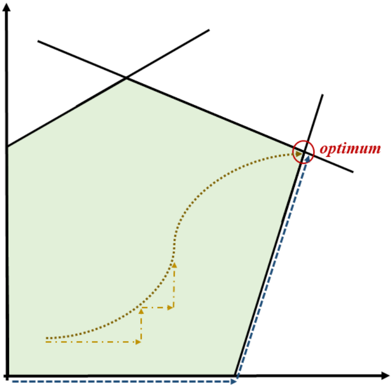
    else given the LP relaxation's optimizer, $\vec{\xi}$, if it contains a *fractional* decision variable ξ_i, construct 2 subproblems $\left\{\dot{\mathcal{F}}, \ddot{\mathcal{F}}\right\}$ by imposing either one of the new constraints $x_i \leq \lfloor \xi_i \rfloor$ or $x_i \geq \lceil \xi_i \rceil$ — and add them $\Omega \leftarrow \Omega \cup \left\{\dot{\mathcal{F}}, \ddot{\mathcal{F}}\right\}$

    /* selection rules needed if #fractional $\xi_i > 2$*/

  end

end

output: $\vec{x}^*$



MP in practice

obtaining an LP standard form

- LP's **standard form** (minimization, equality constraints, non-negative variables):

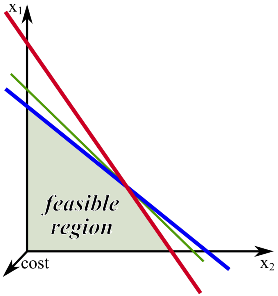
$$\begin{aligned} &\text{minimize}_{\vec{x}}\ \vec{c}^T\vec{x} \\ &\text{subject to: } \mathbf{A}\vec{x} = \vec{b} \\ &\qquad\qquad\quad \vec{x} \geq 0 \end{aligned}$$

- Applicable transformations to obtain standard form (introducing *slack/surplus* variables and accounting for *unrestricted* variables):

$$\begin{array}{lll} \text{(a)} & \max\ \vec{c}^T\vec{x} & \Leftrightarrow \quad -\min\ \left(-\vec{c}^T\vec{x}\right) \\ \text{(b)} & \vec{a}_i^T\vec{x} \leq b_i & \Leftrightarrow \quad \vec{a}_i^T\vec{x} + s_i = b_i,\ s_i \geq 0 \\ \text{(c)} & \vec{a}_i^T\vec{x} \geq b_i & \Leftrightarrow \quad \vec{a}_i^T\vec{x} - s_i = b_i,\ s_i \geq 0 \\ \text{(d)} & -\infty < x_j < \infty & \Leftrightarrow \quad x_j := x_j^+ - x_j^-,\ \ x_j^+ \geq 0,\ x_j^- \geq 0 \end{array}$$

linear programming: solutions

$$\text{minimize } -x_1 - x_2$$
$$\text{subject to: } x_1 + 2x_2 \leq 3$$
$$2x_1 + x_2 \leq 3$$
$$x_1, x_2 \geq 0$$




```
dvar float+ x1,x2,s1,s2;
minimize
   -x1 - x2;
subject to {
    x1 + 2x2 + s1 == 3;
    2x1 + x2 + s2 == 3;
}
```

the fractional (continuous) knapsack problem

n items to be picked in a fractional way, $i = 1, \ldots, n$:

- v_i: value of each item
- w_i: weight of each item

Target: **maximize the total value** within a knapsack of capacity C.

$$\boxed{\begin{array}{l} \textbf{[FKP]} \;\; \text{maximize} \displaystyle\sum_{i=1}^{n} v_i \cdot x_i \\ \text{subject to:} \\ \qquad \displaystyle\sum_{i=1}^{n} x_i \leq C \\ \qquad w_i \geq x_i \in \mathbb{R} \quad \forall i \in 1 \ldots n \end{array}} \tag{7}$$

basic f-knapsack in OPL

```
// Data reading from external database (or sheet or flat file)
{int} N = ...;
{float} CAPACITY = ...;
{float[N]} Values = ...;
{float[N]} Weights = ...;

dvar float+ select_ind[N] in 0..CAPACITY ;

maximize
   sum (n in N) (select_ind[n] * Values[n]) ;

subject to {
      forall (n in N) select_ind[n] <= Weights[n] ;
}
```

integer knapsack in OPL

```
// Data reading from external database
{int} N = ...;
{int} CAPACITY = ...;
{int[N]} Values = ...;
{int[N]} Weights = ...;

dvar int select_ind[N] in 0..1 ;

maximize
   sum (n in N) (select_ind[n] * Values[n]) ;

subject to {
      sum (n in N) select_ind[n]*Weights[n] <= CAPACITY;
}
```

solver operations

- Modern solvers allow the user to choose/tune their core algorithms:

```
cplex.startalg = 1; //primal simplex; for LP relaxation
cplex.lpmethod = 2; //dual simplex
cplex.epgap = 0.001; //relative MIP optimality gap
cplex.IntSolLim = 100; //number of integer solutions to stop
cplex.polishtime = 1800; //polishing time; see text below
cplex.tilim = 1800; //computation time limit
```

- Some MILP solvers actually employ *evolutionary operators* in their heuristic components, such as CPLEX's `polish` subroutine [8].

quadratic programming (QP)

- The simplest formulation of a QP has a *quadratic* objective function and *linear* constraints:

$$\boxed{\begin{array}{ll} \text{minimize}_{\vec{x}} & \frac{1}{2}\vec{x}^T\mathbf{Q}\vec{x} + \vec{c}^T\vec{x} \\ \text{subject to:} & \mathbf{A}\vec{x} \leq \vec{b} \\ & \vec{\ell} \leq \vec{x} \leq \vec{u} \end{array}} \tag{8}$$

- Renowned QP: the Markowitz portfolio – minimizing risk while ensuring minimal ROI, subject to a bounded portfolio investment:

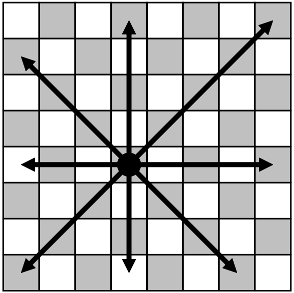
$$\begin{array}{l} \mathbf{Q}: \text{ portfolio's covariance matrix, representing RISK} \\ \vec{c} = \vec{0} \\ \vec{\rho}: \text{ stochastic return, representing ROI} \\ \text{constraints:} \quad \vec{\rho}^T\vec{x} \geq \text{ROI}_{min} \\ \qquad\qquad\quad \sum_i x_i = \text{INVEST}_{total} \end{array} \tag{9}$$

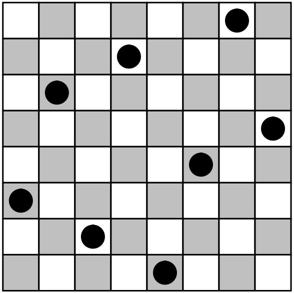
Markowitz: OPL implementation

```
{string} Investments = ...;
float Return[Investments] = ...;
float Covariance[Investments][Investments] = ...;
float BUDGET = ...;
float alpha = ...;

range float FloatRange = 0..BUDGET;
dvar float Allocation[Investments] in FloatRange;

maximize (sum(i in Investments) Return[i]*Allocation[i])
    - alpha*(sum(i,j in Investments)
        Covariance[i][j]*Allocation[i]*Allocation[j]);

subject to {
  // SPEND-IT-ALL: sum of allocations equals the given budget
  allocate: (sum (i in Investments) (Allocation[i])) == BUDGET;
}
```

CSP: the N-queens problem

The N-queens problem (NQP) [9] is defined as the task to place N queens on an $N \times N$ chessboard in such a way that they cannot *attack* each other.





N-queens as maximization

maximize $\sum_{i,j} x_{ij}$

subject to:

$$\sum_i x_{ij} \leq 1 \quad \forall j \in \{1\ldots,N\}$$

$$\sum_j x_{ij} \leq 1 \quad \forall i \in \{1\ldots,N\}$$

$$\sum_{j-i=k} x_{ij} \leq 1 \quad \forall k \in \{-N+2,-N+3,\ldots,N-3,N-2\}$$

$$\sum_{i+j=\ell} x_{ij} \leq 1 \quad \forall \ell \in \{3,4,\ldots,2N-3,2N-1\}$$

$$x_{ij} \in \{0,1\} \;\; \forall i,j \in \{1,\ldots,N\}$$

N-queens: OPL implementation [CSP]

```
int N = ...;
range R = 1..N;

dvar boolean queen [R][R];
// NO OBJECTIVE FUNCTION !
subject to {
  forall (s in R) {
      sum (t in R) queen[s][t] == 1;
     sum (t in R) queen[t][s] == 1;
   }
  forall (k in (-N+2)..(N-2)) {
      sum(s1 in R, t1 in R: t1-s1==k) queen[s1][t1] <= 1;
  }
  forall (k in 3..(2*N-1)) {
      sum(s1 in R, t1 in R: s1+t1==k) queen[s1][t1] <= 1;
  }
}
```

the traveling salesman problem

- The *archetypical* Traveling Salesman Problem (TSP) is posed as finding a Hamilton cycle of minimal total cost. Explicitly, given a directed graph G, with a vertex set $V = \{1, \ldots, d\}$ and an edge set $E = \{\langle i, j \rangle\}$, each edge has cost information $c_{ij} \in \mathbb{R}^+$.

- **Black-box formulation: cyclic permutations**

$$\boxed{\begin{array}{l} \textbf{[TSP-perm]} \;\; \text{minimize} \displaystyle\sum_{i=0}^{d-1} c_{\pi(i),\pi((i+1)_{\text{mod}d})} \\ \text{subject to:} \\ \quad \pi \in P_\pi^{(d)} \end{array}} \qquad (10)$$

- But this is clearly not an MP, since it does not adhere to the canonical form!

ILP formulation [Miller-Tucker-Zemlin]

TSP as an ILP utilizes d^2 binary decision variables $\mathtt{x}_{ij}$:

$$\boxed{\begin{array}{l} \textbf{[TSP-ILP]} \;\; \text{minimize} \displaystyle\sum_{\langle i,j\rangle\in E} c_{ij}\cdot \mathtt{x}_{ij} \\ \text{subject to:} \\ \qquad \displaystyle\sum_{j\in V} \mathtt{x}_{ij} = 1 \quad \forall i \in V \\ \qquad \displaystyle\sum_{i\in V} \mathtt{x}_{ij} = 1 \quad \forall j \in V \\ \qquad \mathtt{x}_{ij} \in \{0,1\} \quad \forall i,j \in V \end{array}} \tag{11}$$

But is this enough? What about inner-circles?

ILP formulation [Miller-Tucker-Zemlin]

TSP as an ILP utilizes d^2 binary decision variables $\mathtt{x}_{ij}$:

$$\begin{aligned}
&\textbf{[TSP-ILP]} \quad \text{minimize} \sum_{\langle i,j\rangle \in E} c_{ij} \cdot \mathtt{x}_{ij} \\
&\text{subject to:} \\
&\qquad \sum_{j \in V} \mathtt{x}_{ij} = 1 \quad \forall i \in V \\
&\qquad \sum_{i \in V} \mathtt{x}_{ij} = 1 \quad \forall j \in V \\
&\qquad \mathtt{x}_{ij} \in \{0,1\} \quad \forall i,j \in V
\end{aligned} \tag{11}$$

But is this enough? What about inner-circles?

d integers $\mathtt{u}_i$ are needed as decision variables to prevent inner-circles:

$$\begin{aligned}
&\ldots \\
&\qquad \mathtt{u}_i - \mathtt{u}_j + 1 \leq (d-1)(1-\mathtt{x}_{ij}) \quad \forall i,j \in 1\ldots d \\
&\qquad d \geq \mathtt{u}_i \geq 2 \quad \forall i \in \{2,3,\ldots,d\}
\end{aligned} \tag{12}$$

the EC perspective

- Unlike GAs, which require dedicated mutation and crossover operators for cyclic permutations, the challenge here is mostly about obtaining an effective formulation
- Perhaps *counter-intuitively*, increasing the order of magnitude of constraints does not necessarily render the problem harder to be solved as MP.
- The given MTZ formulation for TSP is itself of a polynomial size; an alternative formulation possesses $\mathcal{O}\left(2^d\right)$ *subtour elimination constraints*, though **impractical for large graphs**.
- In any case, TSP's *integral hull* is unknown; an NP-hard problem.

- Note that EC researchers have started looking at TSP and other problems in a gray-box perspective: Darrell Whitley's tutorial on "Graybox Optimization and Next Generation Genetic Algorithms".

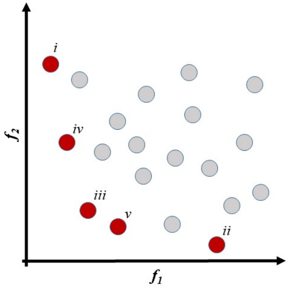
TSP on undirected graphs: OPL implementation

Addressing the undirected TSP by means of "node labeling" – assuming a single visit per node:

```
// Data preparation
tuple Raw_Edge {int point1; int point2; int dist; int active;}
{Raw_Edge} raw_edges = ...;

//Every edge is taken in both directions due to the graph
    nature, using 'union':
tuple Edge {int point1; int point2; int dist;}
{Edge} edges = {<e.point1, e.point2, e.dist> | e in raw_edges :
    e.active == 1}
      union {<e.point2, e.point1, e.dist> | e in raw_edges :
          e.active == 1};
{int} points = {e.point1 | e in edges};
int d = card (points); //set cardinality, i.e., number of cities
```

TSP in OPL continued: core model

```
dvar int edge_selector[edges] in 0..1;
dvar int label[points] in 0..d-1;

minimize sum (e in edges) edge_selector[e]*e.dist;

subject to {
  forall (p in points)
  ct_in_deg_equal_one:
    sum (e in edges : e.point2 == p) edge_selector[e] == 1;
  forall (p in points)
  ct_out_deg_equal_one:
    sum (e in edges : e.point1 == p)edge_selector[e] == 1;
  forall (e in edges : e.point2 != 1)
  ct_monotone_labeling:
    edge_selector [e] == 1 => label [e.point1] ==
        label[e.point2]-1;
}
```



extended topic: multiobjective optimization

multiobjective exact optimization

Diversity Maximization Approach (DMA) [10] key features:

- Iterative-exact nature: obtains a new **exact non-dominated solution** per each iteration
- Criteria exist for the attainment of the complete Pareto frontier
- Fine distribution of the existing set already found is guaranteed
- Optimality gap is provided – what may be gained by continuing constructing the Pareto frontier
- Solves any type of frontier (even if seems as a weighted sum)
- Importantly, DMA is **MILP if the original problem is MILP**

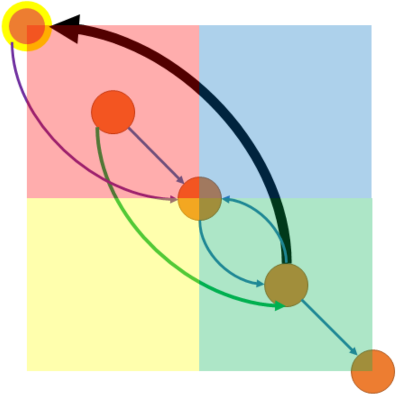
M. Masin and Y. Bukchin, 2008, "Diversity Maximization Approach for Multi-Objective Optimization", *Operations Research*, 56, 411-424.

"high-level" DMA for M-objectives linear problems

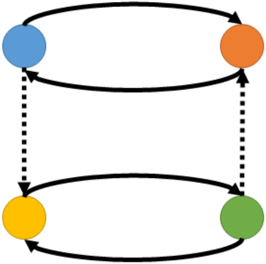
input : a linear program featuring M objectives

1. Find an optimal solution for a weighted sum of multiple objectives with any reasonable strictly positive weights. If there is no feasible solution – **Stop**.
2. Set the partial efficient frontier equal to the found optimal solution. Choose optimality gap tolerance and maximal number of iterations.
3. If the maximal number of iterations is reached – **Stop**, otherwise **add** M **binary variables and** $(M+1)$ **linear constraints to the previous MILP model**.
4. Maximize the proposed diversity measure. If the diversity measure is less than the optimality gap tolerance – **Stop**, otherwise add the optimal solution to the partial efficient frontier and go to Step 3.

output: Pareto set, Pareto frontier



live demo



discussion

quick summary

- MP is a well-established domain encompassing a variety of algorithms with underlying rigorous theory.
- Broad knowledge of MP is valuable for both EC theoreticians and practitioners
- Given convex problems, MP is most likely the fittest tool
- Given discrete optimization problems that may be formulated as MILP/MIQP – it makes sense to first try MP-solvers
- MP is inherently adjusted to constrained problems (unlike EC...)
- Effective MP formulation lies in the heart of practical problem-solving
- Robustness to uncertainty, Pareto optimization, and hybridization are solid extensions to classical MP

communities and resources

- INFORMS: The Institute for Operations Research and the Management Sciences; `https://www.informs.org/`

- COIN-OR: Computational Infrastructure for Operations Research – a project that aims to "create for mathematical software what the open literature is for mathematical theory"; `https://www.coin-or.org/`

- MATHEURISTICS: model-based metaheuristics, exploiting MP in a metaheuristic framework; `http://mh2018.sciencesconf.org/`

partial list of languages and solvers

- Modeling languages:
  1. GAMS
  2. AMPL
  3. OPL
  4. (`python` (Gurobi-Python, SciPy), MATLAB, ...)
- Environments and modeling systems:
  1. OR-Tools — Google Developers (open source!)
  2. IBM ILOG CPLEX (academia-free)
  3. Gurobi
  4. sas
  5. YALMIP
- Third-party solvers (free and open-source):
  1. CBC (via Coin-OR)
  2. GLPK (GNU Linear Programming Kit)
  3. SoPlex
  4. LP_SOLVE

benchmarking and competitions

- MIPLIB: the Mixed Integer Programming LIBrary
  `http://miplib.zib.de/`
- CSPLib: a problem library for constraints
  `http://csplib.org/`
- SAT-LIB: the Satisfiability Library - Benchmark Problems
  `http://www.cs.ubc.ca/~hoos/SATLIB/benchm.html`
- TSP-LIB: the Traveling Salesman Problem sample instances
  `http://comopt.ifi.uni-heidelberg.de/software/TSPLIB95/`

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